

IDENTITIES OF SOME SPECIAL MIXED-TYPE POLYNOMIALS

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ABSTRACT. In this paper, we consider various special mixed-type polynomials which are related to Bernoulli, Euler, Changhee and Daehee polynomials. From those polynomials, we derive some interesting and new identities.

1. INTRODUCTION

Let p be a fixed odd prime number. Throughout this paper, \mathbb{Z}_p , \mathbb{Q}_p and \mathbb{C}_p will denote the ring of p -adic integers, the field of p -adic rational numbers and the completion of the algebraic closure of \mathbb{Q}_p . Let ν_p be the normalized exponential valuation of \mathbb{C}_p with $|p|_p = \frac{1}{p} = p^{-\nu_p(p)}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on \mathbb{Z}_p .

For $f \in UD(\mathbb{Z}_p)$, the bosonic p -adic integral is given by

$$(1) \quad I_0(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad (\text{see [10]}),$$

and the fermionic p -adic integral on \mathbb{Z}_p is defined by Kim to be

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) (-1)^x, \quad (\text{see [12]}).$$

In [7, 8], the higher-order Daehee polynomials are defined by

$$(2) \quad \left(\frac{\log(1+t)}{t} \right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x) \frac{t^n}{n!}, \quad (r \in \mathbb{N}).$$

When $x = 0$, $D_n^{(r)}(0) = D_n^{(r)}$ are called the Daehee numbers of order r .

When $r = 1$, $D_n^{(1)}(x) = D_n(x)$ are called the Daehee polynomials (see [7]).

As is known, the Changhee polynomials of order $s \in \mathbb{N}$ are defined by the generating function to be

$$(3) \quad \left(\frac{2}{t+2} \right)^s (1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(s)}(x) \frac{t^n}{n!}, \quad (\text{see [9]}).$$

When $x = 0$, $Ch_n^{(s)}(0) = Ch_n^{(s)}$ are called the Changhee numbers of order s .

For $s = 1$, $Ch_n^{(1)}(x) = Ch_n(x)$ are called the Changhee polynomials.

The Bernoulli polynomials of order $r \in \mathbb{N}$ are given by

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$$(4) \quad \left(\frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [10, 11, 13–21]}).$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the Bernoulli numbers of order r .

For $r = 1$, $B_n^{(1)}(x) = B_n(x)$ are called the ordinary Bernoulli polynomials.

We recall that the Euler polynomials of order r are defined by the generating function to be

$$(5) \quad \left(\frac{2}{e^t + 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [1–12]}).$$

When $x = 0$, $E_n^{(r)} = E_n^{(r)}(0)$ are called the Euler numbers of order r .

For $r = 1$, $E_n^{(1)}(x) = E_n(x)$ are called the ordinary Euler polynomials.

Finally, the Cauchy polynomials of the first kind of order r are given by

$$(6) \quad \left(\frac{t}{\log(1+t)} \right)^r (1+t)^x = \sum_{n=0}^{\infty} C_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [3, 6]}).$$

When $x = 0$, $C_n^{(r)} = C_n^{(r)}(0)$ are called the Cauchy numbers of the first kind of order r .

For $r = 1$, $C_n^{(1)}(x) = C_n(x)$ are called the ordinary Cauchy polynomials of the first kind (see [3]).

From (1) and (2), we have

$$(7) \quad I_0(f_1) - I_0(f) = f'(0)$$

and

$$(8) \quad I_{-1}(f_1) = -I_{-1}(f) + 2f(0),$$

where $f_1(x) = f(x+1)$ (see [11, 12]).

In this paper, we consider several special polynomials which are derived from the bosonic or fermionic p -adic integral on \mathbb{Z}_p .

Finally, we give some relation or identities of those polynomials.

2. SOME SPECIAL MIXED-TYPE POLYNOMIALS

In this section, we assume that $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. From (7), we can derive the following equation :

$$(9) \quad \begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{(x_1+\cdots+x_r+x)} d\mu_0(x_1) \cdots d\mu_0(x_r) \\ &= \left(\frac{\log(1+t)}{t} \right)^r (1+t)^x = \left(\frac{\log(1+t)}{e^{\log(1+t)} - 1} \right)^r e^{x \log(1+t)} \\ &= \sum_{m=0}^{\infty} B_m^{(r)}(x) \frac{(\log(1+t))^m}{m!} = \sum_{m=0}^{\infty} B_m^{(r)}(x) \sum_{n=m}^{\infty} S_1(n, m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n B_m^{(r)}(x) S_1(n, m) \right) \frac{t^n}{n!}, \end{aligned}$$

and

$$(10) \quad \left(\frac{\log(1+t)}{t} \right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x) \frac{t^n}{n!}.$$

Therefore, by (9) and (10), we obtain the following equation :

$$(11) \quad \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{x_1 + \cdots + x_r + x}{n} d\mu_0(x_1) \cdots d\mu_0(x_r) \\ = \frac{D_n^{(r)}(x)}{n!} = \frac{1}{n!} \sum_{m=0}^n B_m^{(r)}(x) S_1(n, m)$$

where $S_1(n, m)$ is the Stirling number of the first kind.

From (8), we have

$$(12) \quad \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{x_1 + \cdots + x_r + x} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ = \left(\frac{2}{t+2} \right)^r (1+t)^x = \left(\frac{2}{e^{\log(1+t)} + 1} \right)^r e^{x \log(1+t)} \\ = \sum_{m=0}^{\infty} E_m^{(r)}(x) \frac{(\log(1+t))^m}{m!} = \sum_{m=0}^{\infty} E_m^{(r)}(x) \sum_{n=m}^{\infty} S_1(n, m) \frac{t^n}{n!} \\ = \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^n E_m^{(r)}(x) S_1(n, m) \right\} \frac{t^n}{n!}$$

and

$$(13) \quad \left(\frac{2}{t+2} \right)^r (1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(r)}(x) \frac{t^n}{n!}.$$

From (12) and (13)

$$(14) \quad \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{x_1 + \cdots + x_r + x}{n} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ = \frac{Ch_n^{(r)}(x)}{n!} = \frac{1}{n!} \sum_{m=0}^n E_m^{(r)}(x) S_1(n, m).$$

Note that

$$(15) \quad (1+t)^x = \left(\frac{t}{\log(1+t)} \right)^r (1+t)^x \left(\frac{\log(1+t)}{t} \right)^r \\ = \left(\sum_{l=0}^{\infty} C_l^{(r)}(x) \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} D_m^{(r)} \frac{t^m}{m!} \right) \\ = \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} C_l^{(r)}(x) D_{n-l}^{(r)} \right) \frac{t^n}{n!}$$

and

$$(16) \quad (1+t)^x = \sum_{n=0}^{\infty} (x)_n \frac{t^n}{n!}.$$

From (15) and (16), we have

$$(17) \quad \begin{aligned} (x)_n &= \sum_{l=0}^n \binom{n}{l} C_l^{(r)}(x) D_{n-l}^{(r)} \\ &= \sum_{l=0}^n \binom{n}{l} D_{n-l}^{(r)}(x) C_l^{(r)}. \end{aligned}$$

That is,

$$\binom{x}{n} = \frac{1}{n!} \sum_{l=0}^n \binom{n}{l} C_l^{(r)}(x) D_{n-l}^{(r)}.$$

Let us consider the Bernoulli-Euler mixed-type polynomials of order (r, s) as follows :

$$(18) \quad BE_n^{(r,s)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} E_n^{(s)}(x + y_1 + \cdots + y_r) d\mu_0(y_1) \cdots d\mu_0(y_r).$$

Then, we can find the generating function of $BE_n^{(r,s)}(x)$ as follows :

$$(19) \quad \begin{aligned} &\sum_{n=0}^{\infty} BE_n^{(r,s)}(x) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} E_n^{(s)}(x + y_1 + \cdots + y_r) \frac{t^n}{n!} d\mu_0(y_1) \cdots d\mu_0(y_r) \\ &= \left(\frac{2}{e^t + 1} \right)^s \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x+y_1+\cdots+y_r)t} d\mu_0(y_1) \cdots d\mu_0(y_r) \\ &= \left(\frac{2}{e^t + 1} \right)^s \left(\frac{t}{e^t - 1} \right)^r e^{xt}. \end{aligned}$$

Note that

$$(20) \quad \begin{aligned} \left(\frac{2}{e^t + 1} \right)^s \left(\frac{t}{e^t - 1} \right)^r e^{xt} &= \left(\sum_{l=0}^{\infty} E_l^{(s)} \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} B_m^{(r)}(x) \frac{t^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} E_l^{(s)} B_{n-l}^{(r)}(x) \right) \frac{t^n}{n!}. \end{aligned}$$

From (19) and (20), we have

$$(21) \quad BE_n^{(r,s)}(x) = \sum_{l=0}^n \binom{n}{l} E_l^{(s)} B_{n-l}^{(r)}(x).$$

By replacing t by $\log(1+t)$, we get

$$(22) \quad \begin{aligned} &\sum_{n=0}^{\infty} BE_n^{(r,s)}(x) \frac{(\log(1+t))^n}{n!} \\ &= \left(\frac{2}{t+2} \right)^s \left(\frac{\log(1+t)}{t} \right)^r (1+t)^x \\ &= \left(\sum_{l=0}^{\infty} Ch_l^{(s)} \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} D_m^{(r)}(x) \frac{t^m}{m!} \right) \end{aligned}$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^n \binom{n}{m} D_m^{(r)}(x) Ch_{n-m}^{(s)} \right\} \frac{t^n}{n!},$$

and

$$(23) \quad \sum_{m=0}^{\infty} BE_m^{(r,s)}(x) \frac{(\log(1+t))^m}{m!} = \sum_{m=0}^{\infty} BE_m^{(r,s)}(x) \sum_{n=m}^{\infty} S_1(n, m) \frac{t^n}{n!} \\ = \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^n BE_m^{(r,s)}(x) S_1(n, m) \right\} \frac{t^n}{n!}.$$

Therefore, by (22) and (23), we obtain the following equation :

$$(24) \quad \sum_{m=0}^n \binom{n}{m} D_m^{(r)}(x) Ch_{n-m}^{(s)} = \sum_{m=0}^n BE_m^{(r,s)}(x) S_1(n, m).$$

Let us consider the Daehee-Changhee mixed-type polynomials of order (r, s) as follows :

$$(25) \quad DC_n^{(r,s)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} D_n^{(r)}(x + y_1 + \cdots + y_s) d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_s),$$

where $n \geq 0$.

From (25), we can derive the generating function of $DC_n^{(r,s)}(x)$ as follows :

$$(26) \quad \sum_{n=0}^{\infty} DC_n^{(r,s)}(x) \frac{t^n}{n!} \\ = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} D_n^{(r)}(x + y_1 + \cdots + y_s) \frac{t^n}{n!} d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_s) \\ = \left(\frac{\log(1+t)}{t} \right)^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{x+y_1+\cdots+y_s} d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_s) \\ = \left(\frac{\log(1+t)}{t} \right)^r \left(\frac{2}{t+2} \right)^s (1+t)^x.$$

We observe that

$$(27) \quad \left(\frac{2}{t+2} \right)^s \left(\frac{\log(1+t)}{t} \right)^r (1+t)^x = \left(\sum_{l=0}^{\infty} Ch_l^{(s)} \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} D_m^{(r)}(x) \frac{t^m}{m!} \right) \\ = \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^n \binom{n}{m} D_m^{(r)}(x) Ch_{n-m}^{(s)} \right\} \frac{t^n}{n!}.$$

From (26) and (27), we have

$$(28) \quad DC_n^{(r,s)}(x) = \sum_{m=0}^n \binom{n}{m} D_m^{(r)}(x) Ch_{n-m}^{(s)},$$

where $n \geq 0, r, s \in \mathbb{N}$.

Now, we define the Cauchy-Daehee mixed-type polynomials of order (r, s) as follows :

$$(29) \quad CD_n^{(r,s)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} C_n^{(r)}(x + y_1 + \cdots + y_s) d\mu_0(y_1) \cdots d\mu_0(y_s).$$

From (29), we can derive the generating function of $CD_n^{(r,s)}(x)$ as follows :

$$\begin{aligned}
 (30) \quad & \sum_{n=0}^{\infty} CD_n^{(r,s)}(x) \frac{t^n}{n!} \\
 &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} C_n^{(r)}(x + y_1 + \cdots + y_s) \frac{t^n}{n!} d\mu_0(y_1) \cdots d\mu_0(y_s) \\
 &= \left(\frac{t}{\log(1+t)} \right)^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{x+y_1+\cdots+y_s} d\mu_0(y_1) \cdots d\mu_0(y_s) \\
 &= \left(\frac{t}{\log(1+t)} \right)^r \left(\frac{\log(1+t)}{t} \right)^s (1+t)^x. \\
 &= \begin{cases} \sum_{n=0}^{\infty} C_n^{(r-s)}(x) \frac{t^n}{n!} & \text{if } r > s \\ \sum_{n=0}^{\infty} D_n^{(s-r)}(x) \frac{t^n}{n!} & \text{if } r < s \\ \sum_{n=0}^{\infty} (x)_n \frac{t^n}{n!} & \text{if } r = s. \end{cases}
 \end{aligned}$$

Thus, by (30), we get

$$(31) \quad CD_n^{(r,s)}(x) = \begin{cases} C_n^{(r-s)}(x) & \text{if } r > s \\ D_n^{(s-r)}(x) & \text{if } r < s \\ (x)_n & \text{if } r = s \end{cases}$$

where $n \geq 0$.

By replacing t by $e^t - 1$ in (26), we get

$$\begin{aligned}
 (32) \quad & \sum_{n=0}^{\infty} DC_n^{(r,s)}(x) \frac{(e^t - 1)^n}{n!} = \left(\frac{t}{e^t - 1} \right)^r e^{xt} \left(\frac{2}{e^t + 1} \right)^s \\
 &= \left(\sum_{n=0}^{\infty} B_l^{(r)}(x) \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} E_m^{(s)} \frac{t^m}{m!} \right) \\
 &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} B_l^{(r)}(x) E_{n-l} \right) \frac{t^n}{n!},
 \end{aligned}$$

and

$$\begin{aligned}
 (33) \quad & \sum_{m=0}^{\infty} DC_m^{(r,s)}(x) \frac{(e^t - 1)^m}{m!} = \sum_{m=0}^{\infty} DC_m^{(r,s)}(x) \sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n DC_m^{(r,s)}(x) S_2(n, m) \right) \frac{t^n}{n!}.
 \end{aligned}$$

Therefore, by (32) and (33), we get

$$(34) \quad \sum_{m=0}^n DC_m^{(r,s)}(x) S_2(m, n) = \sum_{l=0}^n \binom{n}{l} B_l^{(r)}(x) E_{n-l},$$

where $S_2(n, m)$ is the Stirling number of the second kind.

Finally, we consider the Cauchy-Changhee mixed-type polynomials of order (r, s) as follows :

$$(35) \quad CC_n^{(r,s)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} C_n^{(r)}(x + y_1 + \cdots + y_s) d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_s),$$

where $n \geq 0$.

By (35), we see that the generating function of $CC_n^{(r,s)}(x)$ are given by

$$\begin{aligned}
 (36) \quad & \sum_{n=0}^{\infty} CC_n^{(r,s)}(x) \frac{t^n}{n!} \\
 &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} C_n^{(r)}(x + y_1 + \cdots + y_s) \frac{t^n}{n!} d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_s) \\
 &= \left(\frac{t}{\log(1+t)} \right)^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{x+y_1+\cdots+y_s} d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_s) \\
 &= \left(\frac{t}{\log(1+t)} \right)^r \left(\frac{2}{t+2} \right)^s (1+t)^x \\
 &= \left(\sum_{m=0}^{\infty} C_m^{(r)}(x) \frac{t^m}{m!} \right) \left(\sum_{l=0}^{\infty} Ch_l^{(s)} \frac{t^l}{l!} \right) \\
 &= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^n \binom{n}{m} C_m^{(r)}(x) Ch_{n-m}^{(s)} \right\} \frac{t^n}{n!}.
 \end{aligned}$$

Thus, by (36), we get

$$(37) \quad CC_n^{(r,s)}(x) = \sum_{m=0}^n \binom{n}{m} C_m^{(r)}(x) Ch_{n-m}^{(s)}.$$

By replacing t by $e^t - 1$, we get

$$\begin{aligned}
 (38) \quad & \sum_{n=0}^{\infty} CC_n^{(r,s)}(x) \frac{(e^t - 1)^n}{n!} = \left(\frac{e^t - 1}{t} \right)^r \left(\frac{2}{e^t + 1} \right)^s e^{xt} \\
 &= \left(\sum_{l=0}^{\infty} \frac{S_2(l+r, l)}{\binom{l+r}{r}} \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} E_m^{(s)}(x) \frac{t^m}{m!} \right) \\
 &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \frac{S_2(l+r, l)}{\binom{l+r}{r}} E_{n-l}^{(s)}(x) \binom{n}{l} \right) \frac{t^n}{n!},
 \end{aligned}$$

and

$$\begin{aligned}
 (39) \quad & \sum_{l=0}^{\infty} CC_l^{(r,s)}(x) \frac{(e^t - 1)^l}{l!} = \sum_{l=0}^{\infty} CC_l^{(r,s)}(x) \sum_{n=l}^{\infty} S_2(n, l) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n CC_l^{(r,s)}(x) S_2(n, l) \right) \frac{t^n}{n!}.
 \end{aligned}$$

Therefore, by (38) and (39), we obtain the following identities.

$$\sum_{l=0}^n CC_l^{(r,s)}(x) S_2(n, l) = \sum_{l=0}^n \frac{\binom{n}{l}}{\binom{l+r}{r}} S_2(l+r, l) E_{n-l}^{(s)}(x),$$

where $n \geq 0$.

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